JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **86**, 425–441 (1982)

Initial Values for Parabolic Functions

H. S. BEAR

Department of Mathematics, University of Hawaii, 2565 The Mall, Honolulu, Hawaii 96822

Submitted by K. Fan

1. SMOOTH KERNELS

In [2] we showed that if $\{K_\alpha\}$ is a smooth approximate identity (definition below) and $f \in L_1(-\infty, \infty)$, then $f * K_\alpha \to f$ a.e. The condition that $f \in L_1$ is unduly restrictive. In this section we first explicate what is actually proved in [2], and then (Theorem 9) give a useful sufficient condition on f which insures that $f * K_\alpha \to f$ a.e. when K_α consists of translates of the heat kernel. In the second and third sections of the paper, we consider in further detail the approximate identity given by the heat kernel

$$k_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}, \tag{1}$$

and by its translates

$$k_{(\eta,t)}(x) = \frac{1}{\sqrt{4\pi t}} e^{-(x-\eta)^2/4t}.$$

In Section 2 we obtain new results on the relationship between functions parabolic in a strip $0 < t < T$ and their initial values. In Section 3 we show that parabolic functions can have initial values which grow arbitrarily rapidly.

All integrals without limits are to be interpreted as integrals from $-\infty$ to $+\infty$.

Let $\{K_\alpha\}$ be a net of non-negative functions on $(-\infty, \infty)$, where α runs over some directed set D, and we write $\alpha \to \infty$ to indicate limits as α runs over D. Thus we write, for example, $K_\alpha(x_0) \to 0$ as $\alpha \to \infty$.

The net $\{K_\alpha\}$ is an *approximate identity* (*positive kernel*) if (a), (b), and (c) below are satisfied, and a *smooth approximate identity*, or *smooth kernel*, if, in addition, (d) and (e) are satisfied.

(a) $K_\alpha \in L_1$ and $K_\alpha \geqslant 0$ for all α.

(b) $\int K_\alpha = 1$ for all α.

0022-247X/82/040425-17\$02.00/0

(c) $\int_{|x| \leqslant \delta} K_\alpha \to 1$ as $\alpha \to \infty$, for all $\delta > 0$.

(d) K'_α is continuous on $(-\infty, \infty)$. K_α increases to a unique maximum at x_α, and decreases for $x \geqslant x_\alpha$.

(e) For some constant A, $|x_\alpha| K_\alpha(x_\alpha) \leqslant A$ for all α.

The following lemmas are simple consequences of (a)–(e) above, and are proved in [2].

LEMMA 1. $\int_{|x| \geqslant \delta} K_\alpha \to 0$ *as* $\alpha \to \infty$, *for all* $\delta > 0$.

LEMMA 2. $x_\alpha \to 0$ *as* $\alpha \to \infty$.

LEMMA 3. $\sup\{K_\alpha(s): |s| \geqslant \delta\} \to 0$ *as* $\alpha \to \infty$, *for all* $\delta > 0$. *In particular*, $K_\alpha(s_0) \to 0$ *as* $\alpha \to \infty$ *for all* $s_0 \neq 0$.

LEMMA 4. $K_\alpha(s) \to 0$ *as* $|s| \to \infty$ *for all* α.

LEMMA 5. *For all* $\delta > 0$, $\int_{|x| \geqslant \delta} K'_\alpha \to 0$ *as* $\alpha \to \infty$.

LEMMA 6. *There is a constant B and* $\alpha_0 \in D$ *such that* $\int |xK'_\alpha(x)| \, dx \leqslant B$ *for all* $\alpha > \alpha_0$.

In addition to the properties of $\{K_\alpha\}$, we need the following form of the fact that the derivative of an integral is the integrand.

LEMMA 7. *If f is locally integrable, then for almost all x (and all x where f is continuous),*

$$\lim_{s \to x} \frac{1}{s-x} \int_x^s (f(u) - f(x)) \, du = 0. \tag{2}$$

We will say that *f is in the domain of* $\{K_\alpha\}$, written $f \in \operatorname{dom}\{K_\alpha\}$, iff f is a locally integrable function such that $\int f(s) K_\alpha(x-s) \, ds$ exists as a conditionally convergent improper integral for all x and all $\alpha > \alpha_0$; and for all x and all $\delta > 0$,

$$\int_{-\infty}^{x-\delta} f(s) K_\alpha(x-s) \, ds \to 0 \qquad \text{as} \quad \alpha \to \infty, \tag{3}$$

and

$$\int_{x+\delta}^{\infty} f(s) K_\alpha(x-s) \, ds \to 0 \qquad \text{as} \quad \alpha \to \infty. \tag{4}$$

If $f \in L_1$, then of course (3) and (4) are automatic. However, we want to consider functions f which are unbounded at ∞ but nevertheless are in the domain of the kernel.

THEOREM 8. *If* $\{K_\alpha\}$ *is a smooth approximate identity, and* $f \in \operatorname{dom}\{K_\alpha\}$, *then* $(f * K_\alpha)(x) \to f(x)$ *a.e. (and in particular for all* x *where* f *is continuous) as* $\alpha \to \infty$.

Proof. In [2, Theorem 1, p. 20] we proved this theorem for $f \in L_1$. In fact the proof uses only the fact that f is locally integrable (as in Lemma 7 above) and the limits (3) and (4) above hold. For the reader's convenience, we outline the proof. Fix x such that (2) holds, and let $\delta > 0$,

$$\begin{aligned} f * K_\alpha(x) - f(x) &= \int_{-\infty}^{\infty} K_\alpha(x-s)(f(s)-f(x))\, ds \\ &= \int_{-\infty}^{x-\delta} K_\alpha(x-s) f(s)\, ds - f(x) \int_{-\infty}^{x-\delta} K_\alpha(x-s)\, ds \\ &\quad + \int_{x+\delta}^{\infty} K_\alpha(x-s) f(s)\, ds - f(x) \int_{x+\delta}^{\infty} K_\alpha(x-s)\, ds \\ &\quad + \int_{x-\delta}^{x} K_\alpha(x-s)(f(s)-f(x))\, ds \\ &\quad + \int_{x}^{x+\delta} K_\alpha(x-s)(f(s)-f(x))\, ds. \end{aligned} \tag{5}$$

The second and fourth integrals above tend to zero with α by Lemma 1. The first and third integrals tend to zero by conditions (3) and (4); i.e., the assumption that $f \in \operatorname{dom}\{K_\alpha\}$. The last two integrals are similar, and we treat $J_5 = \int_{x-\delta}^{x}$. Let

$$\beta(s) = \int_x^s (f(s) - f(x))\, ds,$$

so that $\beta(s)/(s-x) \to 0$ as $s \to x$ by (2). Then

$$\begin{aligned} J_5 &= \int_{x-\delta}^{x} K_\alpha(x-s)\, d\beta(s) \\ &= K_\alpha(0)\, \beta(x) - K_\alpha(\delta)\, \beta(x-\delta) + \int_{x-\delta}^{x} \beta(s)\, K_\alpha'(x-s)\, ds. \end{aligned}$$

Observe that $\beta(x) = 0$, that β is continuous, and that $K_\alpha(\delta) \to 0$ for any fixed $\delta > 0$. We estimate the final integral as follows:

$$\int_{x-\delta}^{x} \beta(s)\,K_\alpha(x-s)\,ds \leqslant \int_{x-\delta}^{s} \left|\frac{\beta(s)}{s-x}\right| |(x-s)\,K'_\alpha(x-s)|\,ds$$
$$\leqslant \max_{x-\delta\leqslant s\leqslant x} \left|\frac{\beta(s)}{s-x}\right| \cdot \int_0^\delta |tK'_\alpha(t)|\,dt \qquad (6)$$
$$\leqslant B \max \left|\frac{\beta(s)}{s-x}\right|,$$

where B is the constant of Lemma 6. Choose δ so the right side of (6) is less than ε. Then pick α_0 so $K_\alpha(\delta)\,\beta(x-\delta) < \varepsilon$ for $\alpha > \alpha_0$. Hence $|J_5| < 2\varepsilon$ if δ is small enough and $\alpha > \alpha_0$. We can pick δ and α_0 so that the last integral in (5) is also small for $\alpha > \alpha_0$. Now δ is fixed and we pick $\alpha_1 > \alpha_0$ so that the first four integrals are small if $\alpha > \alpha_1$.

The kernels we shall be interested in are the heat kernels (1). [See Widder [12, p. 30] for general properties.] If $u(x, t) = (g * k_t)(x)$ with $g \in \operatorname{dom}\{k_t\}$, then Theorem 8 says that $\lim_{t\to 0+} u(x, t) = g(x)$ a.e. To inquire about convergence along paths other than vertical, we consider a new approximate identity consisting of translates of $\{k_t\}$. Let

$$k_{(\eta,t)}(x) = k_t(x-\eta).$$

In [2, p. 24] we showed that $\{k_{(\eta,t)}\}$ is again a smooth approximate identity as $t \to 0+$, $\eta \to 0$, provided $t \geqslant A\eta^2$ for some $A > 0$. The condition $t \geqslant A\eta^2$ insures that (e) holds, and (a), (b), (c), (d) are obvious. If $u(x, t) = (g * k_t)(x)$, then $(g * k_{(\eta,t)})(x) = u(x-\eta, t)$ and we conclude that $u(x_0-\eta, t) \to g(x_0)$ a.e. as $t \to 0+$ with $t \geqslant A\eta^2$. This is the same as saying that $u(x, t) \to g(x_0)$ a.e. as $(x, t) \to (x_0, 0)$ over any parabola $t = A(x-x_0)^2$. (This result is due originally to Hattemer [5].)

The following theorem gives a sufficient condition that a locally bounded measurable function G be in the domain of the general heat kernel $\{k_{(\eta,t)}\}$.

THEOREM 9. *If G is a locally bounded measurable function such that $\int |G(s)|\, e^{-as^2}\, ds = M < \infty$ for some $a > 0$, then $G \in \operatorname{dom}\{k_{(\eta,t)}\}$.*

Proof. If $t < 1/4a$, then $k_{(\eta,t)}(x-s)\,G(s)$ is absolutely integrable for all x and η, and only conditions (3) and (4) are at issue. We check (4):

$$\int_{x+\delta}^{\infty} |G(s)|\, e^{-as^2} \left[e^{as^2} \frac{1}{\sqrt{4\pi t}}\, e^{-(x-\eta-s)^2/4t}\right] ds$$
$$\leqslant M \sup_{s\geqslant x+\delta} \left[\frac{e^{as^2}}{\sqrt{4\pi t}} \exp\{-(x-\eta-s)^2/4t\}\right].$$

The term in brackets on the right above can be written by completing the square:

$$\exp\left[\frac{a(x-\eta)^2}{1-4at}\right]\cdot\frac{1}{\sqrt{4\pi t}}\exp\left\{-\left(\frac{1-4at}{4t}\right)\left(s-\frac{x-\eta}{1-4at}\right)^2\right\}. \quad (7)$$

For fixed x, the left factor remains bounded as $t\to 0+$, $\eta\to 0$. If t and η are small enough, then

$$\frac{1-4at}{4t}\geqslant 1/8t$$

and for $s\geqslant x+\delta$,

$$\left|s-\frac{x-\eta}{1-4at}\right|\geqslant p>0$$

for some $p>0$. Hence the right-hand factor of (7) is less than

$$\frac{1}{\sqrt{4\pi t}}e^{-p^2/8t},$$

which tends to zero as $t\to 0+$. The proof of condition (3) is similar, so $G\in\operatorname{dom}\{k_{(\eta,t)}\}$. Notice that we did not *here* use the condition $t\geqslant A\eta^2$, but only the fact that both t and η approach 0. However, the condition $t\geqslant A\eta^2$ is necessary that $\{k_{(\eta,t)}\}$ be a smooth kernel, which gives us the constant B of Lemma 6, which is used in the proof of Theorem 8.

We will need the following corollary in the next section.

COROLLARY. *If G is a locally bounded measurable function such that $|G(s)|\leqslant Me^{cs^2}$ for some M, some $c>0$, then $G\in\operatorname{dom}\{k_{(\eta,t)}\}$.*

Proof. If $a>c$, then $G(s)\,e^{-as^2}$ is integrable.

2. Initial Values of Parabolic Functions

We will say that $u(x,t)$ is *parabolic* in a domain provided u satisfies the heat equation, $u_{xx}=u_t$, and the partial derivatives are continuous up to order two. It follows [10, p. 201] that u is infinitely differentiable in t, and analytic in x. We will be particularly interested in functions parabolic in a half-plane $t>0$, or a strip $0<t<T$.

Parabolic functions, like harmonic functions, can be parabolic in a half-plane and have zero boundary values without being identically zero. However, a function cannot be harmonic in the whole plane (and therefore analytic in each variable) and identically zero in a half-plane without vanishing identically. A parabolic function, on the other hand, can vanish identically for $t \leqslant t_0$ without being identically zero. The possibility that a parabolic function can bifurcate along any line $t = t_0$ is a fact which must be reckoned with in relating parabolic functions and their initial values. The following result, although not spelled out, is implicit in Tychonoff's paper [10, p. 200]. (See also [8, p. 609].)

THEOREM 10. *There is a non-zero function $w(x, t)$, parabolic in the whole plane, with $w(x, t) \equiv 0$ for $t \leqslant t_0$.*

Proof. Following Tychonoff we invoke the following result of Carleman [3, p. 63]: if $\{A_n\}$ is an increasing sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{A_n}} < \infty, \tag{8}$$

then there is a C^∞ function $F(t)$ on $(-\infty, \infty)$ such that $F^{(n)}(0) = 0$ for all n, and $|F^{(n)}(t)| \leqslant A_n$ for all n, t. If $1 < k < 2$, then $A_n = (kn)!$ is such a sequence. To see this, we recall Stirling's formula [1, p. 204, Ex. 2]

$$x! = \left(\frac{x}{e}\right)^x \sqrt{x}\sqrt{2\pi}\exp(\theta(x)/12), \tag{9}$$

where $2x/(2x + 1) < \theta(x) < 1$. Hence

$$\sqrt[n]{(kn)!} \geqslant \left(\frac{kn}{e}\right)^k = \left(\frac{k}{e}\right)^k n^k,$$

and

$$\sum_{n=1}^{\infty} 1/\sqrt[n]{(kn)!} \leqslant \left(\frac{e}{k}\right)^k \sum_{n=1}^{\infty} n^{-k} < \infty.$$

Now let $F(t)$ be a C^∞ function on $(-\infty, \infty)$ with $F^{(n)}(0) = 0$, $|F^{(n)}(t)| \leqslant (kn)!$, $1 < k < 2$, and let

$$u(x, t) = \sum_{n=0}^{\infty} F^{(n)}(t) \frac{x^{2n}}{(2n)!}. \tag{10}$$

The series obtained by differentiating (10) $2m$ times with respect to x, or m times with respect to t, is

$$\sum_{n=0}^{\infty} F^{(n+m)}(t) \frac{x^{2n}}{(2n)!}. \tag{11}$$

If a_n is the nth term, then using Stirling's formula and letting E denote the exponential terms which tend to one as $n \to \infty$, we have

$$
\begin{aligned}
|a_n|^{1/n} &\leqslant \left[\frac{x^{2n}}{(2n)!}(k(n+m))!\right]^{1/n} \\
&= x^2 \left[\left(\frac{k(n+m)}{e}\right)^{k(n+m)} \left(\frac{e}{2n}\right)^{2n} \sqrt{\frac{k(n+m)}{2n}} \cdot E\right]^{1/n} \\
&= x^2 e^{2-k-(km)/n} \left(\frac{k(n+m)}{2n}\right)^{k+km/n} \\
&\quad \times \left(\frac{k(m+n)}{2n}\right)^{1/2n} E^{1/n} \cdot \frac{1}{(2n)^{2-k-km/n}}.
\end{aligned}
$$

As $n \to \infty$, the first five terms approach $x^2 \cdot e^{2-k} \cdot (k/2)^k \cdot 1 \cdot 1$, and the last term approaches zero. Hence $\lim |a_n|^{1/n} = 0$, and the limit is uniform for $-x_0 \leqslant x \leqslant x_0$, for any x_0. Therefore (10) can be differentiated term-by-term infinitely often with respect to x or t. It follows that $u(x, t)$ is parabolic for all (x, t), that $u(x, 0) \equiv 0$, and that all t-derivatives of $u(x, t)$ are zero at all points $(x, 0)$. Hence if $v(x, t) = u(x, t)$ for $t > 0$ and $v(x, t) \equiv 0$ for $t \leqslant 0$, then v is parabolic in the whole plane. The function $w(x, t) = v(x, t - t_0)$ is the desired function.

Tychonoff's uniqueness theorem for parabolic functions says that if $u(x, t)$ is parabolic in some strip, $u(x, t) \to 0$ as $(x, t) \to (x_0, 0)$, for each x_0, and $|u(x, t)| \leqslant Me^{ax^2}$ for some M, some a, then $u(x, t) \equiv 0$. The exponential growth condition is therefore sufficient to keep a parabolic function from bifurcating along some line $t = t_0 \geqslant 0$. Tychonoff's proof, without change, shows that in fact one need only assume $|u(x, t)| \leqslant Me^{ax^2}/\sqrt{t}$, and we will need this slightly stronger result.

THEOREM 11. *If $u(x, t)$ is parabolic for $0 < t < T$ and $u(x, t) \to 0$ as $(x, t) \to (x_0, 0)$ for each x_0, and $|u(x, t)| < Me^{ax^2}/\sqrt{t}$ for some M, some a, then $u(x, t) \equiv 0$.*

Proof. [10, p. 206; 11, p. 88].

In accordance with Theorem 11 we will say that $u(x, t)$ is *monolithic* on the strip $0 < t < T$ provided u is parabolic in the strip, and for each $c < T$, there are M and a such that

$$|u(x, t)| \leqslant Me^{ax^2}/\sqrt{t} \tag{12}$$

for all x and t, $0 < t \leqslant c$. Two such functions which agree along any line $t = t_0$ $(0 \leqslant t_0 < T)$ must then agree for $t \geqslant t_0$. At $t_0 = 0$, "u and v agree" means $\lim_{(x,t) \to (x_0,0)}(u(x, t) - v(x, t)) = 0$ for all x_0.

If φ is a locally bounded measurable function, we define its Poisson integral as

$$P(\varphi) = P(\varphi; x, t) = \lim_{M,N\to\infty} \int_{-M}^{N} \varphi(s)\, k_t(x-s)\, ds, \tag{13}$$

where $k_t(x)$ is the heat kernel (1). Observe that the Poisson integral (13) is conditionally convergent by definition. Tychonoff showed [10, pp. 201, 202] that if (13) converges for soe (x_0, t_0), then (13) converges for all (x, t) with $0 < t < t_0$, and $P(\varphi; x, t)$ is parabolic for $0 < t < t_0$.

We will say that a function $u(x, t)$ parabolic on the strip $0 < t < T$ has the *semigroup property*, or *Huygens property* (cf. [13]), if

$$u(x, t+h) = \int u(s, h)\, k_t(x-s)\, ds \tag{14}$$

whenever $t > 0$, $h > 0$, and $t + h < T$. The semigroup property (14) has the same basic importance for parabolic functions that the local Poisson integral does for harmonic functions. One sees from Theorem 10 that the semigroup property is not automatic for parabolic functions.

THEOREM 12. *If φ is a locally bounded measurable function such that*

$$\int |\varphi(s)|\, e^{-as^2}\, ds = M_0 < \infty \tag{15}$$

for some $a > 0$, and $u(x, t) = P(\varphi; x, t)$, then $u(x, t)$ is monolithic for $0 < t < 1/4a$, and $u(x, t) \to \varphi(x_0, 0)$ a.e. as $(x, t) \to (x_0, 0)$ over parabolas.

Proof. If $t < 1/4a$, then $k_t(x-s) \leqslant Ke^{-as^2}$ for some constant K depending on x and t; hence $P(\varphi; x, t)$ converges absolutely and $u(x, t)$ is parabolic. Theorem 9 shows that $\varphi \in \operatorname{dom}\{k_{(\eta,t)}\}$, so $u(x, t)$ approaches $\varphi(x_0)$ a.e. over parabolas. To see that $u(x, t)$ is monolithic, we estimate the integral $P(\varphi)$ as follows (cf. [10, p. 208] and (7) above):

$$\begin{aligned} |u(x,t)| &\leqslant \int |\varphi(s)|\, e^{-as^2} \frac{1}{\sqrt{4\pi t}} e^{-[(x-s)^2 - 4ats^2]/4t}\, ds \\ &= \frac{1}{\sqrt{4\pi t}} \exp\left[\frac{ax^2}{1-4at}\right] \int |\varphi(s)|\, e^{-as^2} e^{N(s)}\, ds, \end{aligned} \tag{16}$$

where

$$N(s) = -\frac{1}{4t}\left[s\sqrt{1-4at} - \frac{x}{\sqrt{1-4at}}\right]^2 \leqslant 0.$$

The identity (16) is simply an algebraic identity in the exponent. Hence from (15) and (16) we have

$$|u(x,t)| \leqslant \frac{M_0}{\sqrt{4\pi t}} e^{ax^2/(1-4at)}, \tag{17}$$

which is the monolithicity condition (12) with $M = M_0/\sqrt{4\pi}$ and a replaced by $a/(1-4ac)$ for $0 < t \leqslant c < 1/4a$.

The following result is a refinement of Lemma 6.2 of [9].

THEOREM 13. *If $u(x,t)$ is monolithic with*

$$|u(x,t)| \leqslant Me^{ax^2}/\sqrt{t}$$

for $0 < t < 1/4a$, then $u(x,t)$ has the semigroup property for $0 < t < 1/4a$.

Proof. Let $v(x,t) = P(u(s,t_0); x, t)$. With $u(s,t_0)$ for φ in (15) and

$$\frac{M}{\sqrt{t_0}} \int e^{-\epsilon s^2}\, ds$$

for M_0, we conclude that $v(x,t)$ is monolithic for $0 < t < 1/4(a+\varepsilon)$ for any $\varepsilon > 0$. This is the same as saying $v(x,t)$ is monolithic for $0 < t < 1/4a$. We also have $v(x,t) \to u(x_0,t_0)$ as $(x,t) \to (x_0,0)$, since $u(\cdot,t_0)$ is continuous and $\{k_t\}$ is an approximate identity (here "smooth" is irrelevant). Hence both u and v are monolithic and agree along the line $t = t_0$ in the sense that $v(x,0) = u(x,t_0)$. Therefore $v(x,t) = u(x,t_0+t)$ for $0 < t_0 + t < 1/4a$, and this is the semigroup property for $u(x,t)$.

LEMMA 14. *Let $u(x,t)$ be parabolic for $0 < t < 1/4a$ and let $u_t(x) = u(x,t)$. If the functions $u_t(x)\, e^{-ax^2}$ have bounded L_p norms $(1 \leqslant p \leqslant \infty)$ for $0 < t < 1/4a$, then $u(x,t)$ has the semigroup property for $0 < t < 1/4a$ if $p = \infty$, and for $0 < t < 1/4ap$ if $1 \leqslant p < \infty$.*

Proof. The condition that the L_1 norms be bounded is sufficient by [13, Theorem 2.3; 12, p. 162]. If $\|u_t(x)\, e^{-ax^2}\|_\infty \leqslant M$, then $\|u_t(x)\, e^{-a'x^2}\|_1 \leqslant M'$ for each $a' > a$, and hence $u(x,t)$ has the semigroup property on $0 < t < 1/4a'$ for each $a' > a$, which is the same as having the semigroup property on $0 < t < 1/4a$. Now assume that the L_p norms are bounded,

$$\int |u(x,t)|^p\, e^{-apx^2}\, dx \leqslant M^p$$

for $0 < t < 1/4a$. Since $\sqrt{ap/\pi}\, e^{-apx^2}\, dx$ is a measure of total mass one, and $u_t(x)$ is integrable with respect to this measure, we have Jensen's inequality

$$\left[\int |u(x, t)| \sqrt{ap/\pi}\, e^{-apx^2}\, dx\right]^p$$

$$\leqslant \int |u(x, t)|^p \sqrt{ap/\pi}\, e^{-apx^2}\, dx$$

$$\leqslant \sqrt{ap/\pi}\, M^p.$$

Hence

$$\int |u(x, t)|\, e^{-apx^2}\, dx \leqslant (ap/\pi)^{1/2p - 1/2} M,$$

and by Theorems 12 and 13, $u(x, t)$ has the semigroup property for $0 < t < 1/4ap$.

THEOREM 15. *Let $u(x, t)$ be parabolic for $0 < t < 1/4a$;*

(i) *if* $\|u_t(x)\, e^{-ax^2}\|_\infty \leqslant M$ *for* $0 < t < T \leqslant 1/4a$, *then* $u(x, t) = P(e^{as^2} g(s))$ *with* $\|g\|_\infty \leqslant M$;

(ii) *if* $\|u_t(x)\, e^{-ax^2}\|_1 \leqslant M$ *for* $0 < t < T \leqslant 1/4a$, *then* $u(x, t) = P(e^{as^2}\, d\mu(s))$, *where μ is a signed Borel measure with* $\|\mu\| \leqslant M$;

(iii) *if* $\|u_t(x)\, e^{-ax^2} - g(x)\|_1 \to 0$ *as* $t \to 0+$ *for some* $g \in L_1$, *then* $u(x, t) = P(e^{as^2} g(s))$;

(iv) *if* $\|u_t(x)\, e^{-ax^2}\|_p \leqslant M$ *for* $0 < t < T \leqslant 1/4a$, *then* $u(x, t) = P(e^{as^2} g(s))$ *for some g with* $\|g\|_p \leqslant M$.

Notes. Some of the parts of Theorem 15 are known. We nevertheless include them here because the same sort of w^*-compactness arguments are involved in all the proofs. These arguments are patterned after [7, pp. 18, 23], where they are applied to the Fejér kernel and the Poisson kernel, and give analogous results for harmonic functions. Part (i) is given in [6, Theorem 13.2]. Theorem 10.2 of [6] and Theorem 8 of [5] are "if and only if" versions of (ii) for the case $a = 0$. Theorem 9.6 of [6] is an "if and only if" version of (iv) for the case $a = 0$. Theorem 2.3 of [13] is (ii), without the estimate on $\|\mu\|$. More important, however, is the fact that Theorem 2.3 of [13] establishes the critical semigroup property, which was also used in our Lemma 14. In the analogous proofs for harmonic functions, the semigroup property is automatic. The principal philosophical content of Theorems 15 and 16 is that $e^{ax^2}\, dx$ is a more natural measure than dx for coupling with heat kernels, and $e^{-ax^2}\, dx$ is more natural than dx for measuring the growth of sections $u_t(x)$ of parabolic functions. Theorem 16 below gives partial converses of (i), (ii), and (iii).

Proof of Theorem 15. (i) If the L_∞ norms of the functions $u_t(x)\, e^{-ax^2}$ are bounded by M, then there is a subnet $u_\tau(x)\, e^{-ax^2}$ which converges w^* (i.e., pointwise on L_1) to some $g \in L_\infty$ with $\|g\|_\infty \leqslant M$. By Lemma 14, $u(x, t)$ has the semigroup property for $t < 1/4a$. Moreover, if $t < 1/4a$, and x is fixed, $e^{as^2} k_t(x-s) \in L_1$. As $\tau \to 0+$, we have

$$u(x, t+\tau) = \int u(s, \tau)\, e^{-as^2} k_t(x-s)\, e^{as^2}\, ds$$

$$\to \int g(s)\, k_t(x-s)\, e^{as^2}\, ds;$$

i.e., $u(x, t) = P(e^{as^2} g(s))$.

(ii) Let $d\mu_t(x) = u_t(x)\, e^{-ax^2}\, dx$, so that for $t < 1/4a$

$$\|\mu_t\| = \int |u_t(x)|\, e^{-ax^2}\, dx \leqslant M.$$

Here $\|\mu_t\|$ is the norm μ_t has in C_0^*, the adjoint space of the space of continuous functions which vanish at infinity. Again by Alaoglu's Theorem there is subnet $\{\mu_\tau\}$ of the measures $\{\mu_t\}$ which converges w^* to some μ, with $\|\mu\| \leqslant M$, as $\tau \to 0+$. For fixed x and t, with $t < 1/4a$, $e^{as^2} k_t(x-s) \in C_0$; hence, as $\tau \to 0+$,

$$u(x, t+\tau) = \int u(s, \tau)\, e^{-as^2} k_t(x-s)\, e^{as^2}\, ds$$

$$\to \int k_t(x-s)\, e^{as^2}\, d\mu(s);$$

i.e., $u(x, t) = P(e^{as^2}\, d\mu(s))$.

(iii) Let $\varphi(x) = e^{ax^2} g(x)$, so the assumption is

$$\int |u(s, \tau)\, e^{-as^2} - \varphi(s)\, e^{-as^2}|\, ds \to 0$$

as $\tau \to 0+$, or

$$\int |u(s, \tau) - \varphi(s)|\, e^{-as^2}\, ds \to 0.$$

For each fixed x and t, with $t < 1/4a$, we have $k_t(x-s) < Ke^{-as^2}$, where K depends only on x and t. Hence

$$\int |u(s,\tau) - \varphi(s)|\, k_t(x-s)\, ds \to 0$$

as $\tau \to 0+$, and consequently

$$u(x, t+\tau) = \int u(s,\tau)\, k_t(x-s)\, ds$$

$$\to \int \varphi(s)\, k_t(x-s)\, ds.$$

That is, $u(x,t) = P(\varphi(s)) = P(e^{as^2} g(s))$.

(iv) Assume $\|u_t(x)\, e^{-ax^2}\|_p \leqslant M$. Since the M-ball of $L_p = L_q^*$ is w^* compact, there is again a subnet $\{u_\tau(x)\, e^{-ax^2}\}$ which converges pointwise on L_q to some $\varphi \in L_p$, with $\|\varphi\|_p \leqslant M$. For fixed x and t, with $t < 1/4a$, $e^{as^2} k_t(x-s) \in L_q$, and as before we get

$$u(x,t) = \int \varphi(s)\, e^{as^2} k_t(x-s)\, ds,$$

or $u(x,t) = P(e^{as^2}\varphi(s))$ with $\varphi \in L_p$, $\|\varphi\|_p \leqslant M$.

THEOREM 16. (i) *If* $u(x,t) = P(e^{as^2} g(s))$ *with* $g \in L_\infty$, *then for all* $b > a$,

$$\limsup_{t\to 0+} \|u_t(x)\, e^{-bx^2}\|_\infty \leqslant \|g\|_\infty .$$

(ii) *If* $u(x,t) = P(e^{as^2}\, d\mu(s))$, *where* μ *is a finite signed measure, then for all* $b > a$,

$$\limsup_{t\to 0+} \|u_t(x)\, e^{-bx^2}\|_1 \leqslant \|\mu\|.$$

(iii) *If* $u(x,t) = P(e^{as^2} g(s))$, *where* $g \in L_1$, *then for each* $b > a$,

$$\limsup_{t\to 0+} \|u_t(x)\, e^{-bx^2}\|_1 \leqslant \|g\|_1 .$$

Proof. (i) From [10, p. 208; 12, p. 171] we have

$$P(e^{as^2}) = \frac{1}{\sqrt{1-4at}}\, e^{ax^2/(1-4at)}, \tag{18}$$

so

$$|u_t(x)| \leqslant \frac{M}{\sqrt{1-4at}} e^{ax^2/(1-4at)} \qquad (M = \|g\|_\infty)$$

and equality holds if $g(s) \equiv M$. If $b > a$, then for all sufficiently small t,

$$b > a/(1-4at)$$

and $\|u_t(x)\, e^{-bx^2}\|_\infty \leqslant M/\sqrt{1-4at}$.

Note that $\|u_t(x)\, e^{-ax^2}\|_\infty = \infty$ for all $t > 0$ if $u(x, t) = P(e^{as^2})$, so the theorem cannot be improved to $\limsup_{t \to 0+} \|u_t(x)\, e^{-ax^2}\|_\infty \leqslant \|g\|_\infty$.

(ii) We assume $u(x, t) = P(e^{as^2}\, d\mu(s))$ and estimate $\|u_t(x)\, e^{-bx^2}\|_1$ as

$$\begin{aligned} \int |u_t(x)|\, e^{-bx^2}\, dx &= \int \left| \int \frac{e^{-bx^2}}{\sqrt{4\pi t}} e^{-(x-s)^2/4t} e^{as^2}\, d\mu(s) \right| dx \\ &\leqslant \frac{1}{\sqrt{4\pi t}} \iint (e^{-bx^2} e^{-(x-s)^2/4t})\, dx\, d|\mu|(s), \end{aligned} \tag{19}$$

where the change of order of integration (for small values of t) will be justified by the finiteness of the final estimate. The inside integral above can be written

$$\begin{aligned} &\frac{e^{-bs^2/(4bt+1)}}{\sqrt{4\pi t}} \int \exp -\frac{1}{4t} \left(\sqrt{4bt+1}\, x - \frac{s}{\sqrt{4bt+1}} \right)^2 dx \\ &= \frac{1}{\sqrt{4bt+1}} e^{-bs^2/(4bt+1)}, \end{aligned} \tag{20}$$

where the integral is evaluated with the substitution

$$u = \frac{1}{\sqrt{4t}} \left(\sqrt{4bt+1}\, x - \frac{s}{\sqrt{4bt+1}} \right),$$

$$\int e^{-u^2}\, du = \sqrt{\pi}.$$

Now we rewrite (19) using (20):

$$\int |u_t(x)|\, e^{-bx^2}\, dx \leqslant \frac{1}{\sqrt{4bt+1}} \int e^{-bs^2/(4bt+1)} e^{as^2} d|\mu|(s). \tag{21}$$

If t is small enough, $b/(4bt+1) > a$, and we have

$$\|u_t(x)\, e^{-bx^2}\|_1 \leqslant \frac{1}{\sqrt{4bt+1}} \|\mu\|,$$

which implies the desired result.

Observe that here also the condition $\limsup \|u_t(x)\, e^{-bx^2}\|_1 \leqslant \|\mu\|$ for $b > a$ cannot be improved to $\limsup \|u_t(x)\, e^{-ax^2}\|_1 \leqslant \|\mu\|$. To see this let $d\mu(x) = dx/(1+x^2)$, so $\|\mu\| = \pi$. The computation above, (21), with equality holding throughout since $\mu \geqslant 0$, shows that with $b = a$ we get

$$\|u_t(x)\, e^{-ax^2}\|_1 = \frac{1}{\sqrt{4at+1}} \int e^{\varepsilon s^2} \frac{1}{1+s^2}\, ds = \infty$$

for all $t > 0$, where $\varepsilon = [a - a/(4at+1)] > 0$.

(iii) If $u(x,t) = P(e^{as^2} g(s))$ with $g \in L_1$, then we can replace $d\mu$ in (ii) with $g(s)\, ds$, so $d\,|\mu|(s) = |g(s)|\, ds$ and the proof is as before.

3. Parabolic Functions with Large Initial Values

We have seen in Theorems 15(i) and 16(i) that parabolic functions $u(x,t)$ which satisfy a condition of the form

$$|u(x,t)| \leqslant M e^{ax^2} \tag{22}$$

are Poisson integrals of functions $\varphi(s)$ which satisfy

$$|\varphi(s)| \leqslant M e^{as^2}. \tag{23}$$

Moreover (Theorem 9), $u(x,t) \to \varphi(x_0)$ a.e. as $(x,t) \to (x_0, 0)$ over parabolas. Conversely, if $u(x,t) = P(e^{as^2} g(s))$ with $\|g\|_\infty = M$, then $\|u(x,t)\, e^{-bx^2}\|_\infty \leqslant M/\sqrt{1-4t}$ for $b > a$, and t small enough. There is therefore a one-to-one correspondence between parabolic functions which satisfy an exponential growth condition (22) and initial values $\varphi(s)$ which satisfy an exponential growth condition (23).

In this section we show that there are functions $g(s)$ on the line such that $|g(s)|$ grows arbitrarily rapidly, but nevertheless $g \in \operatorname{dom}\{k_t\}$. Specifically, we will show there are g, with $|g|$ essentially arbitrary, so that $u(x,t) = P(g)$ extends to a function parabolic for all $t > 0$ and $u(x,t) \to g(x)$ a.e. as $t \to 0+$. The construction involves changing the sign of a rapidly growing function $\varphi(s)$ on adjacent small intervals so that the resulting function $g(s) = \pm\varphi(s)$ has conditionally convergent Poisson integral.

We define an *ε-partition of* $[a, b]$ to be a set of equi-spaced points $\{x_i\}$ with $a = x_0 < x_1 < \cdots < x_n = b$, and $x_i - x_{i-1} \leqslant \varepsilon$, and n an even integer.

LEMMA 17. *Let $\varphi(x)$ be a positive monotone function on $[a, b]$. If $T(x)$ is alternately -1 and $+1$ on the subintervals of an ε-partition of $[a, b]$, then*

$$\left| \int_a^b T(x)\,\varphi(x)\,dx \right| \leqslant \varepsilon \max \varphi(x). \tag{24}$$

Proof. If A_i is the area under $y = \varphi(x)$ over the ith subinterval, then

$$\int_a^b T(x)\,\varphi(x)\,dx = (A_2 - A_1) + (A_4 - A_3) + \cdots + (A_n - A_{n-1}).$$

A picture shows that the areas represented by $A_2 - A_1$, $A_4 - A_3$, etc., can be stacked disjointly in a rectangle of width ε and height $\varphi(b) - \varphi(a) \leqslant \max \varphi(x)$.

LEMMA 18. *If $\varphi(x)$ is any positive piecewise monotone function on $(-\infty, \infty)$ and $\eta > 0$, then there is a partition of $(-\infty, \infty)$ into subintervals so that if $T(x)$ is alternately $+1$ and -1, then $I = \int T(x)\,\varphi(x)\,dx$ exists as an improper Riemann integral, and $|I| < \eta$.*

Proof. Let $\cdots x_{-2} < x_{-1} < x_0 < x_1 < x_2 < \cdots$ be the integers plus the finite number of points where φ changes from monotone increasing to monotone decreasing. On each interval $I_n = [x_{n-1}, x_n]$, $n = 0, \pm 1, \pm 2, \ldots$, put an ε_n-partition, where $\varepsilon_n \max \varphi[I_n] < \eta/(3 \cdot 2^{|n|})$. Then

$$\left| \int_J T(x)\,\varphi(x)\,dx \right| < \eta/(3 \cdot 2^{|n|}) \tag{25}$$

for any interval $J \subset I_n$. Clearly $\int T(x)\,\varphi(x)\,dx$ exists as an improper Riemann integral, and has value numerically less than η.

THEOREM 19. *If the functions $\varphi(s)$ and $\varphi(s)\,k_t(x-s)$ are piecewise monotone for all x and all $t < t_1$, and all have fewer than N turning points, then there is a function $T(s)$ which is alternately $+1$ and -1 on adjacent intervals so that $T(s)\,\varphi(s) \in \operatorname{dom}\{k_t\}$, and consequently $u(x, t) = P(T(s)\,\varphi(s)) \to \pm\varphi(x)$ a.e. as $t \to 0+$. Moreover, $T(s)$ can be chosen so that $u(x, t)$ extends to a function parabolic for all $t > 0$.*

Proof. Choose a partition of $(-\infty, \infty)$ and the corresponding function T as in Lemma 18, for the function φ, so that (25) holds for each subinterval.

If $k_t(x-s) \leqslant 1$, then (25) will still hold with $\varphi(s)$ replaced by $\varphi(s)\,k_t(x-s)$, on any subinterval $[x_n, x_{n+1}]$ on which $\varphi(s)\,k_t(x-s)$ is monotone. If $\varphi(s)\,k_t(x-s)$ has k of its at most N turning points in $[x_n, x_{n+1}]$, then for $x_n \leqslant a < b \leqslant x_{n+1}$,

$$\left| \int_a^b T(x)\,\varphi(s)\,k_t(x-s)\,ds \right| \leqslant k\eta/(3 \cdot 2^{|n|}); \tag{26}$$

i.e., another $\eta/(3 \cdot 2^{|n|})$ is necessary on the right side for each turning point in $[x_n, x_{n+1}]$. In any case, (26) holds, with $k = N$, for any x and t, if $k_t(x-s) \leqslant 1$ on $[x_n, x_{n+1}]$.

Now fix a value of x, and note that

$$\int_{x-1}^{x+1} T(s)\,\varphi(s)\,k_t(x-s)\,ds \tag{27}$$

exists for all t. For $t < t_1$, $k_t(x-s) \leqslant 1$ on $[x+1, \infty)$ and on $(-\infty, x-1]$. Hence, from (26), both integrals

$$\int_{x+1}^{\infty} T(s)\,\varphi(s)\,k_t(x-s)\,ds, \qquad \int_{-\infty}^{x-1} T(s)\,\varphi(s)\,k_t(x-s)\,ds,$$

converge, and uniformly in x and t if $t < t_1$. Hence $P(T(s)\,\varphi(s); x, t)$ converges, and uniformly in x and t, if $t < t_1$.

To complete the demonstration that $T(s)\,\varphi(s) \in \operatorname{dom}\{k_t\}$, we must show that for each fixed x and each $\delta > 0$,

$$\int_{x+\delta}^{\infty} T(s)\,\varphi(s)\,k_t(x-s)\,ds \to 0 \tag{28}$$

at $t \to 0+$, and a similar statement for the integral from $-\infty$ to $x-\delta$. Fix $\delta > 0$ and let $\varepsilon > 0$. Pick $B > x+\delta$ so that

$$\int_B^{\infty} T(s)\,\varphi(s)\,k_t(x-s)\,ds < \varepsilon \tag{29}$$

for all x, t, with $t < t_1$, using (26). Clearly

$$\int_{x+\delta}^{B} T(s)\,\varphi(s)\,k_t(x-s)\,ds \to 0$$

as $t \to 0+$ since $k_t(x-s) \to 0$ uniformly for $x+\delta \leqslant s \leqslant B$. Therefore $|\int_{x+\delta}^{\infty}| < 2\varepsilon$ if $0 < t < t_0 \leqslant t_1$, and we have (28). The companion statement for $\int_{-\infty}^{x-\delta}$ is proved the same way.

For $\tau = 1/4\pi$, $k_\tau(x-s) \leqslant 1$, and a partition and function T can be found so that

$$\left| \int T(s)\,\varphi(s)\,k_\tau(x-s)\,ds \right| \leqslant 1$$

for all x. By taking the common refinement of this partition and that used earlier in the proof and the corresponding T, we see that $T(s)\,\varphi(s) \in \operatorname{dom}\{k_t\}$ and $|u(x,t)| = |P(\varphi(s)\,T(s))| \leqslant 1$ on $t = 1/4\pi$. This $u(x,t)$ can obviously be extended to be parabolic in the whole plane, but of course nothing can be said about monolithicity near $t = 0$.

The condition that all the functions $\varphi(s)\,k_t(x-s)$ are piecewise monotone and have a limited number of turning points is not very restrictive. For example, $\pm\exp s^3$ or $\pm\exp(\exp s^3)$ are possible initial values for a function parabolic on $t > 0$.

References

1. L. V. Ahlfors, "Complex Analysis," 2nd ed., McGraw–Hill, New York, 1966.
2. H. S. Bear, Approximate identities and pointwise convergence, *Pacific J. Math.* **81** (1979), 17–27.
3. T. Carleman, Les fonctions quasi analytiques, Gauthier–Villars, Paris, 1926.
4. F. W. Gehring, On solutions of the equation of heat conduction, *Michigan Math. J.* **5** (1958), 191–202.
5. J. R. Hattemer, Boundary behavior of temperatures, I, *Studia Math.* **25** (1964), 111–155.
6. I. I. Hirschman and D. V. Widder, "The Convolution Transform," Princeton Univ. Press, Princeton, N.J., 1955.
7. K. Hoffman, "Banach Spaces of Analytic Functions," Prentice–Hall, Englewood Cliffs, N.J., 1962.
8. P. C. Rosenbloom and D. V. Widder, A temperature function which vanishes initially, *Amer. Math. Monthly* **65** (1958), 607–609.
9. P. C. Rosenbloom and D. V. Widder, Expansions in terms of heat polynomials and associated functions, *Trans. Amer. Math. Soc.* **92** (1959), 220–266.
10. A. Tychonoff, Théorèmes d'unicité pour l'équation de la chaleur, *Rec. Math.* (*Mat. Sb.*) **42** (1935), 199–216.
11. D. V. Widder, Positive temperatures on an infinite rod, *Trans. Amer. Math. Soc.* **55** (1944), 85–95.
12. D. V. Widder, "The Heat Equation," Academic Press, New York, 1975.
13. D. V. Widder, The Huygens property for the heat equation, *Trans. Amer. Math. Soc.* **232** (1977), 239–244.